Growth in coupled multiplicative stochastic processes

International Workshop
The Physics Approach to Risk:
Agent-Based Models and Networks

Jan Lorenz

Chair of Systems Design

October 27,2008

Outline

1. Swiss COST P10 Project: Failure Risk Propagation in Economic and Supply Networks

2. Growth in Coupled Multiplicative Stochastic Processes

Failure Risk Propagation in Economic and Supply Networks[1]

3 Parts:

1. Fragility dynamics with trend persistence
   - Result: Non-monotonous impact of link density
2. **Growth in coupled multiplicative stochastic processes**
3. Cascades in credit and supply networks
   - In preparation

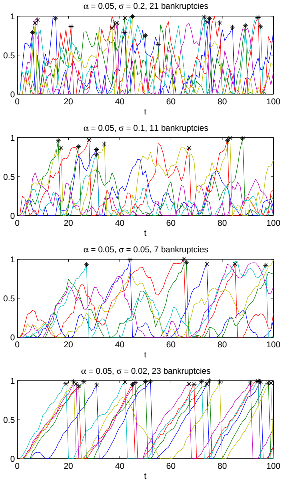
[1] Chair of Systems Design (Prof. Schweitzer, Stefano Battiston) and Institute for Operations Research (Prof. Lüthi, Marco Laumanns), ETH Zurich

Less volatility, higher failure risk?[2]

- Fragility of n firms evolves as

$$\underbrace{\phi(t+1) = \phi(t)}_{\text{fragility}} + \underbrace{\sigma\xi(t)}_{\text{stochastic shocks}} + \underbrace{\alpha \,\mathrm{sign}(\Delta\phi(t))}_{\text{trend reinforcing}}$$

- trend reinforcing $\nearrow \rightsquigarrow \nearrow\nearrow$, $\searrow \rightsquigarrow \searrow\searrow$
- reducing volatility σ
  - decreases stochastic shocks $\rightarrow$ less bankruptcies, BUT
  - reduces possibility to break bad trends $\rightarrow$ more bankrupcies!
- Conclusion: We are safest with intermediate volatility



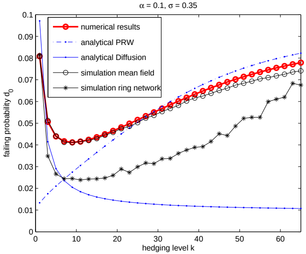
2 Jan Lorenz, Stefano Battiston, Networks and Heterogeneous Media, 2008

Local optimum explained by stochastic process

- Scaling of displacement for Gaussian Random Walk (GRW) and Persistent Random Walk (PRW)

$$\phi(t+1) = \phi(t) + \underbrace{\sigma\xi(t)}_{\text{diffusive scaling}} + \underbrace{\alpha\text{trend}}_{\text{ballistic} \to \text{diffusive}}$$

- GRW dominates for $\frac{\alpha}{\sigma} \to 0$, PRW for $\frac{\alpha}{\sigma} \to \infty$



Outline

1. Swiss COST P10 Project: Failure Risk Propagation in Economic and Supply Networks

2. Growth in Coupled Multiplicative Stochastic Processes

Motivation

Growth is fundamental to economic (and other) development

- Many processes grow proportional to size (**multiplicative growth**)
  - invested money
  - knowledge or innovativeness
  - populations
- Growing entities exchange in a **linear input-output** manner
  - trade
  - exchange of knowledge

What is impact of combining the two on the **effective growth rate**?

In the following:

1. Take **knowledge** as proxy for the dynamic variable $x(t)$
2. Define simple **internal stochastic** multiplicative growth dynamics
3. Define simple linear input-putput dynamics for a small **ensemble**
4. Find: Even when
   - internal dynamics lead to **extinction**
   - input-output is lossy and leads to **shrinking**

   $\rightarrow$ **The combination can imply growth**

5. Phase transitions positive/negative growth rates (**Risk of Extinction!**), regarding e.g.
   - number of firms
   - sharing rate

Multiplicative stochastic knowledge production

- The knowledge of a firm to be intially $x(0) = 1$
- There is a **positive random variable** for the growth rate η
- Dynamics take place in **discrete time** as

$$x(t+1) = \eta(t)x(t)$$

- As example p **success rate**, r **loss rate**, b **gain rate**,

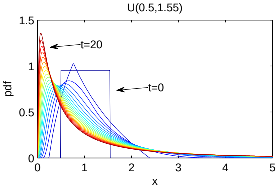
$$\eta_{r,b,p}(t) = \begin{cases} 1+b & \text{with probability } p, \\ 1-r & \text{with probability } 1-p. \end{cases}$$

- **Let's assume**: $\langle \eta_{r,b,p} \rangle > 1$ because of hard work/knowledge management/optimism

Multiplicative stochastic process $x(t) = \eta(t)x(t)$

- In general (for $\log \eta$ with finite variance)
  - The central limit theorem for $\log x(t+1) = \log \eta(t) + \log x(t)$ implies for large t that $x(t)$ is close to **log-normal** distributed, with cdf and pdf

$$F_{x(t)}(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left[\frac{\log(x) - t\mu_{\log \eta}}{\sigma_{\log \eta}\sqrt{t}\sqrt{2}}\right]$$

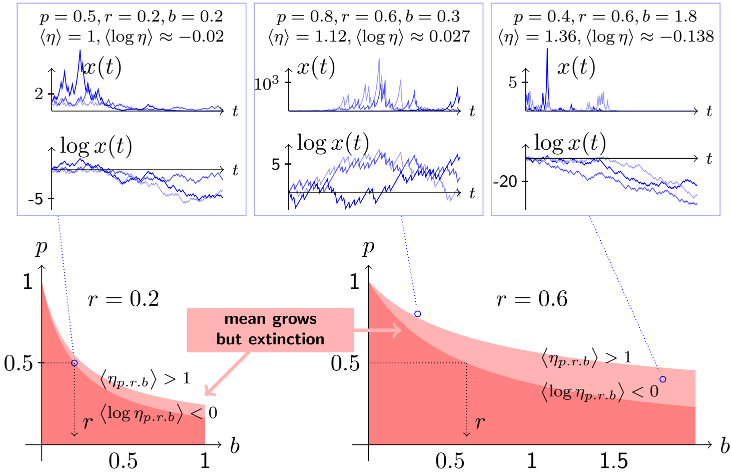


with $\mu_{\log \eta} = \langle \log \eta \rangle$, $\sigma^2_{\log \eta} = \left\langle (\log \eta)^2 \right\rangle - \langle \log \eta \rangle^2$
  - For $\mu_{\log \eta} < 0$: **Extinction!** (every quantile $\to 0$ with $t \to \infty$)
- For $\eta_{p,r,b}$

$$\langle \eta_{p,r,b} \rangle = p(1+b) + (1-p)(1-r)$$

$$\langle \log \eta_{p,r,b} \rangle = p\log(1+b) + (1-p)\log(1-r)$$

Behavior of $x(t+1) = \eta_{p,r,b}x(t)$



Input-output knowledge sharing

- Ensemble of n firms, with knowledge $x(0) \in \mathbb{R}^n_{>0}$
- Nonnegative **input-output matrix** A, a_{ij} is the fraction of knowledge i receives from j
- Dynamics

$$x(t+1) = Ax(t) = A^t x(0)$$

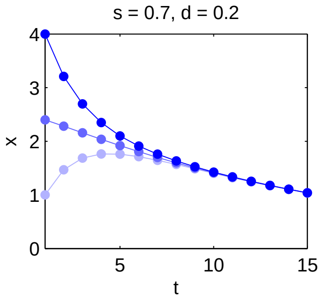
- As example s **keep self rate**, d **loss rate for shared**

$$A_{s,d} = \begin{bmatrix} s & \frac{(1-s)(1-d)}{n-1} & \cdots & \frac{(1-s)(1-d)}{n-1} \\ \frac{(1-s)(1-d)}{n-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{(1-s)(1-d)}{n-1} \\ \frac{(1-s)(1-d)}{n-1} & \cdots & \frac{(1-s)(1-d)}{n-1} & s \end{bmatrix}$$

Lossy knowledge sharing

- For $n = 3$, $s = 0.7$, $d = 0.2$

$$A_{s,d} = \begin{bmatrix} 0.7 & 0.12 & 0.12 \\ 0.12 & 0.7 & 0.12 \\ 0.12 & 0.12 & 0.7 \end{bmatrix},$$



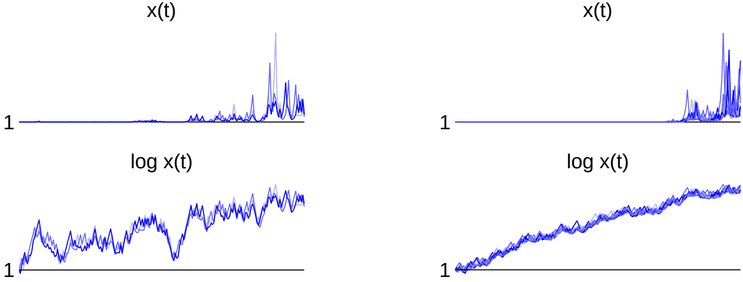
- Spectral radius $\rho(A_{s,d}) = 1 - d(1 - s) = 0.94$
- For $d > 0, s > 0$ Extinction!

Combination 1: Sharing of knowledge

For $s = 0.7, d = 0.2, p = 0.4, r = 0.6, b = 1.8$

- $\eta_{p,r,b}(t)$ now n-dim vector of iid random variables
- Knowledge shared by input-output matrix after production

$$x(t+1) = A_{s,d}\left(\eta_{p,r,b}(t) \bullet x(t)\right)$$



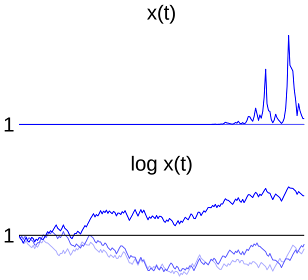
$n = 3$ $n = 10$

Combination 2: Sharing of potential

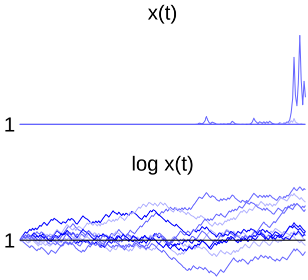
For $s = 0.7, d = 0.2, p = 0.4, r = 0.6, b = 1.8$

- $\eta_{p,r,b}(t)$ now n-dim vector of iid random variables
- Random growth rates are shared by input-output matrix

$$x(t+1) = (A_{s,d}\ \eta_{p,r,b}(t)) \bullet x(t)$$



$n = 3$



$n = 10$

Phase transitions

For $d = 0.2, p = 0.4, r = 0.6, b = 1.8$

- Intermediate **keep self rate** induces growth
- **More firms** can induce growth
- For 'potential': growth rate $= \exp \left\langle \log(s\eta + \sum_{i=1}^{n-1} \frac{(1-s)(1-s)}{n-1}\eta) \right\rangle$