Opinion Dynamics under Heterogeneous Bounds of Confidence for the Agents

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May 2003

Abstract

A group of m agents is to find a common agreement about a certain issue. An opinion is represented by a real number. We assume that each agent changes his opinion by repeated averaging over the opinions of his agents of confidence. An agent i should be an agent of confidence for agent j, if the opinion of agent i lies within the confidence interval of agent j, which is an interval with the opinion of j in the center. This iterated process is called opinion formation under bounded confidence.

In this paper we model heterogenity of agents by admitting differences in the sizes of the confidence intervals for each agent. We examine what happens if some of the agents have higher/lower bounds of confidence than the other agents. We focus on how the possibility of reaching a consensus among all agents changes through heterogenity. Interestingly there are both, positive and negative effects. Simulations show that the positive effects might be bigger.

In addition, we present some mathematical results about the convergence of this kind of opinion formation.

1 The Model

Consider a group of m agents, each having an opinion about a certain issue. An opinion should be a real number. The group of agents is now to find an agreement. We suppose that each agent is willing to revise his opinion by taking the opinions of other competent agents into consideration. A competent agent in the view of one agent should be an agent with an opinion close to his own opinion. Further we suppose that all agents revise their opinions at the same time. The repitition of this simultaneous revising is what we call a process of opinion formation under bounded confidence.

The bounded confidence model goes back to Krause/Hegselmann [4] and Dittmer [3, 2]. In the basic model the bound of confidence is equal for all agents. In this paper we take the first steps in examining the effects, that different bounds of confidence for each agent can have on the dynamic of opinion formation.

In section 2 we show some new phenomena which evolve when some agents have higher or lower bounds of confidence. Then we switch to a more systematical analysis by simulation in section 3. We will analyse opinion dynamic processes in which we assume that the sum of confidence bounds must be equal, but different distributions are possible. Our aim is to see what kinds of heterogenety has positive effects on finding a consensus. In section 4 we present some basic mathematical results.

For $m \in \mathbb{N}$ we define $\underline{m} := \{1, \dots, m\}$. This is our set of agents.

We call $x(t) \in \mathbb{R}^m$ an *opinion profile* of m agents at time step $t \in \mathbb{N}$. For each agent $i \in \underline{m}$ we assume a *bound of confidence* $\epsilon_i > 0$, so we get a *confidence vector* $\epsilon \in \mathbb{R}^m_{>0}$, we can sort this vector, such that $\epsilon_1 \geq \dots \geq \epsilon_m$ (if we permute the number of agents in the same way).

We define the *confidence set* of agent i in an opinion profile $x \in \mathbb{R}^m$ as

$$I(x, i) := \{j \in \underline{m} \,|\, |x_i - x_j| \leq \epsilon_i\}.$$

We call $[x_i - \epsilon_i, x_i + \epsilon_i]$ the *confidence interval* of agent i.

We assume that all agents distribute equal confidence weights between all agents with opinions which lie in their confidence interval. Thus we define a *confidence matrix* $A(x) \in \mathbb{R}^{m \times m}$ for an opinion profile $x \in \mathbb{R}^m$ by

$$A(X)_{[i,j]} := \begin{cases} \frac{1}{\#I(x,i)} & \text{if } j \in I(x, i) \\ 0 & \text{otherwise.} \end{cases}$$

Now we can set up the *process of opinion formation* as a time-discrete dynamical system. Let $x(0) \in \mathbb{R}^m$ be a starting opinion profile. The process of opinion formation is a series of opinion profiles $(x(t))_{t \geq 0}$ recursively defined through

$$x(t + 1) = A(x(t))x(t).$$

In figure 1 we present a simple example with five agents. Each number

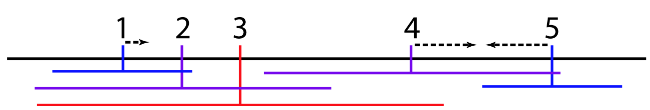

Figure 1: Example with 5 agents

stands for one agent and the coloured lines stand for their confidence intervals. The black line symbolises $\mathbb{R}$. The arrows assign the way the agents will move in the next step of iteration. Red stands for an open-minded agent, blue for a closed-minded and violet for an agent with a middle bound of confidence. The confidence matrix of the opinion profile of figure 1 is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Every process of opinion formation with a uniform bound of confidence for all agents converges to a profile where the set of agents splits into some groups with equal opinions. This has been shown in [6, 7] even when opinions are multidimensional.

This mathematical result is still lacking for processes with different bounds of confidence, although simulations shows the convergence for every starting profile and every bound of confidence. We call the profile $x := lim_{t\to\infty} x(t)$ the *convergence profile*.

2 Phenomena caused by heterogenity

The parameter of most interest in simulation is the bound of confidence. It is examined for the case of a uniform bound in [4, 6, 7]. Here is a quick summary.

Let us start with a starting profile of 50 agents whos opinions are ramdomly distributed in the unit interval $[0, 1]$. Figure 2 shows the dynamic for a uniform range of confidence with $\varepsilon = 0.2$.

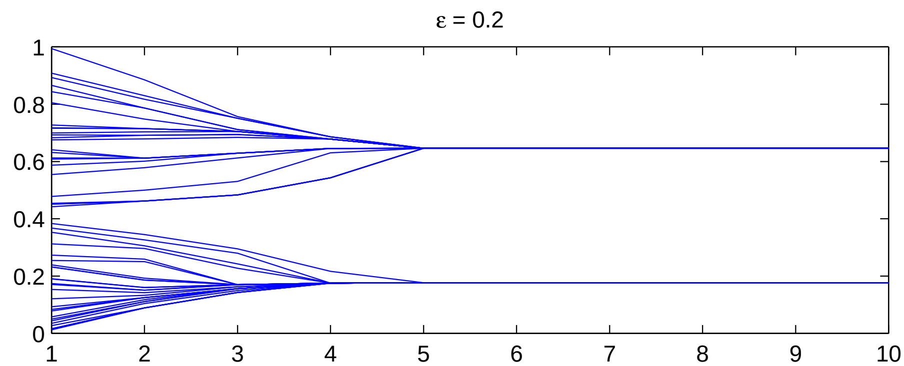

Figure 2: Typical polarization among 50 agents

The process remains stable after time step 5. We see polarization into two groups. This dynamic is typical for a middle bound of confidence. Roughly explained, the polarization occurs in this way: The agents at the extreme points go in direction of the middle, other agents roughly remain in their position. Thus, the diameter of the profile shrinks but the concentration of opinions at the extremes of the profile raises. These higher concentrations attract the agents in the middle of the profile. Thus two groups emerge.

If we lower the bound of confidence, then perhaps one more group in the middle may survive and if we lower the range further, then we got a plurality of opinions in the convergence profile. If we raise the bound of confidence, there is a point at which the two groups are so close to each other, that they unite to a consensus of all agents.

This seems clear by explanation, but this decrease of opinions with rising range of confidence is not monotone because of the complexity of the dynamical behaviour, as shown in [6, 7, 2].

Now we will take a look what may happen to the dynamic of these 50 agents if few of them got higher or lower bounds of confidence. Figure 3 shows four examples.

Explanation of Figure 3 The starting profile is in every case the same as in figure 2. Most of the agents got still a bound of confidence of 0.2. Blue

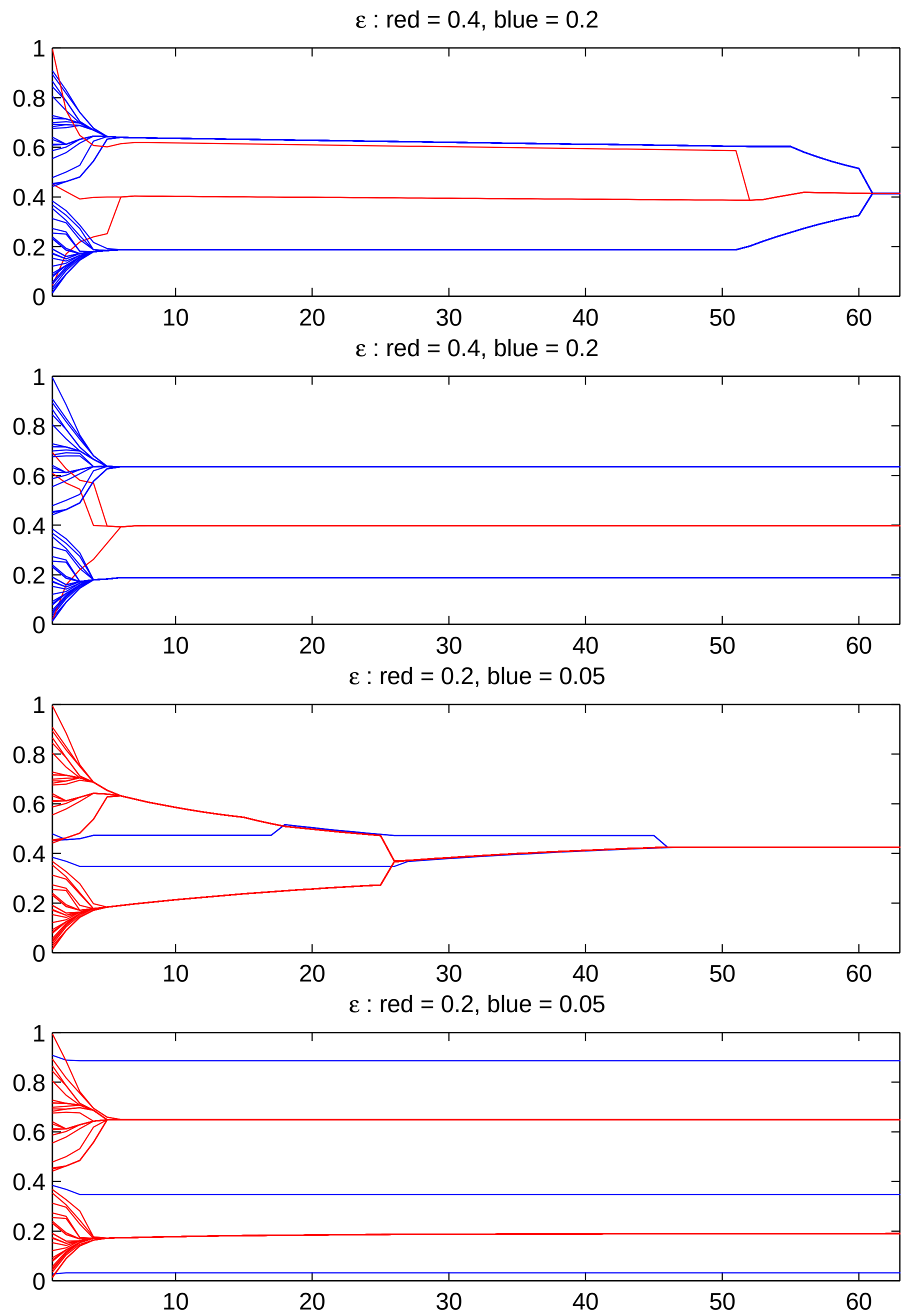

Figure 3: Different phenomena, when three agents got higher/lower ϵ_i

stands for “closed-minded” agents with lower bounds than the others. Red stands for “open-minded” agents with higher bounds than the others.

First: Three agents got a bound of 0.4. The first five time steps are very similar to figure 2. But two of the open-minded get in a positon in the middle, where they trust all other agents. The closed-minded at position 0.2 only trust themself and the others in their group. The most important agent is now the other open-minded agent. He trusts the upper closed-minded group and the two other open-minded. Thus, we get a very slow dynamic, in which all open-minded and the upper closed-minded group succesively lower their opinions. After about 55 time steps they get so close, that both closed-minded groups now trust the three, now united, open-minded in the middle. After a few time steps they get so close, that they all join to a consensus.

Second: Now again three agents got a bound of 0.4, but not the same as above. Now they join quickly together in the center, where they trust every agent, but the closed-minded groups at the extremes do not trust them. Thus, the dynamic gets stuck and we got three different opinions in the convergence profile.

Third: Now three agents with central opinions are closed-minded with a bound of confidence of only 0.05. In the first five time steps two of them join in the upper center and one remains in the lower center. All other agents join in the already known two groups. But each group trusts one of the closed-minded groups in the middle. So they get closer to the middle. But when the upper closed-minded joins nearly the upper open-minded, the open-minded trust also the lower closed-minded group and they pull the upper closed-minded closer to the center. After about 47 time steps they all join to a consensus.

Fourth: Now three other agents are closed-minded. But they all remain single minorities with their opinions. The one at about 0.9 trusts nobody else and is not even trusted by anybody. The two other closed-minded trust also nobody else but the open-minded group below trust both of them and thus can not decide. We get five different opinions in the convergence profile.

We summarize the possible phenomena:

- Open-minded agents can bring two closed-minded groups togehter, but they are also in danger to get between them so that no others trust them.
- Closed-minded agents in the center can attract groups at the extremes to join at the center because they are not so much attracted by the extremes. But closed-minded agents may remain minorities.

If we want to support the finding of a consensus, raising and lowering some bounds of confidence may have either positive or negative effects.

We can imagine various kinds of heterogeneous distributed bounds of confidence. For example few open-minded and many closed minded or a mixture of all values between 0.1 and 0.3. In the next section we make a kind of systematical analysis of a class of confidence vectors.

3 Simulating different kinds of heterogenity

Consider again a situation with 50 agents with opinions between 0 and 1 and a confidence vector $\epsilon = 0.2$. Simulation shows, that if we take enough randomly distributed starting profiles, than more then 80 % of the starting profiles converge to a situation of polarization into two groups. Some profiles converge to consensus and same profiles to plurality.

We want to check now different types of confidence vectors for these 50 agents. At first we must define the types of confidence vectors, we want to compare. Our assumption should be, that in every confidence vector $\epsilon \in \mathbb{R}^{50}_{>0}$ the sum of all bounds of confidence should be the same

$$\sum_{i=1}^{50} \epsilon_i = 50 \cdot 0.2 = 10. \tag{1}$$

This assumption says that there is only a fixed sum of confidence in our 50-agent-world, which may be distributed in different ways. The question for us in the role of God in our world is now, how to distribute the confidence to make it easier for the agents to find a consensus.

This assumption is rather constructed, but it is a good hypothesis in our aim to compare different kinds of heterogenity.

Now we construct a family of confidence vectors. We look at the function

$$f(x) = a\,\mathrm{sign}(x)|x|^p.$$

Figure 4 shows the function for $p = 0, 1, 5$. Let us imagine the 50 agents equally

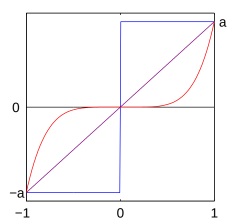

Figure 4: $f(x) = a\,\mathrm{sign}(x)|x|^p$ for $p = 0, 1, 5$ (blue, violet, red)

distributed on the x-axis between -1 and 1, and then give each agent i the function value at his position plus 0.2 as bound of confidence ϵ_i.

In this setting we call a the *range of heterogenity*, and p the *measure of the width of the middle*. For high p we have many agents with medium bounds of confidence and few with high and few with low bounds of confidence. Obviously the confidence vectors constructed like this fulfill (1).

We will do simulations for the (a, p) parameter field with a from 0.01 to 0.2 in steps of 0.01 and $p = 1/10, 1/5, 1/4, 1/3, 1/2, 1, 2, 3, 4, 5, 10$. In figure 5 we see a field with nine examples for confidence vectors.

The dynamic of a process of opinion formation depends very much on the starting profile, thus we choose 250 randomly distributed starting profiles with 50 opinions. And then we compute for each (a, p) parameter constellation the convergence profile. In this paper we are only interested in the number of different opinions in the convergence profiles.

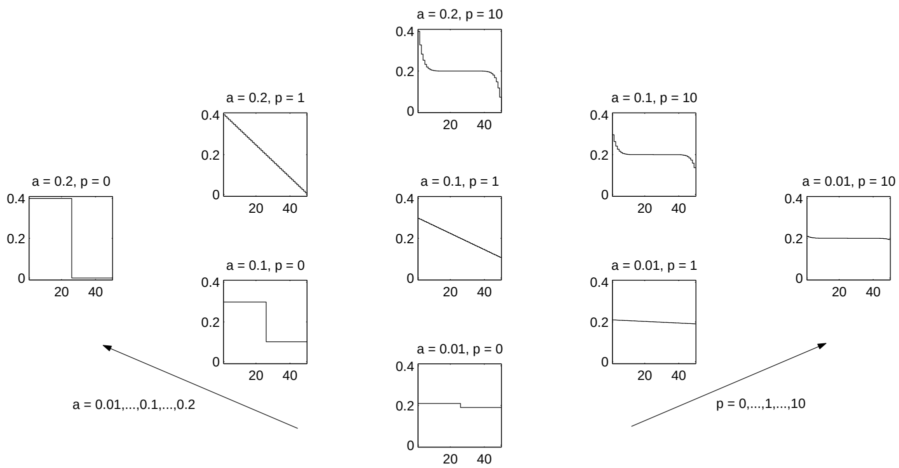

Figure 5: 9 Examples of confidence vectors

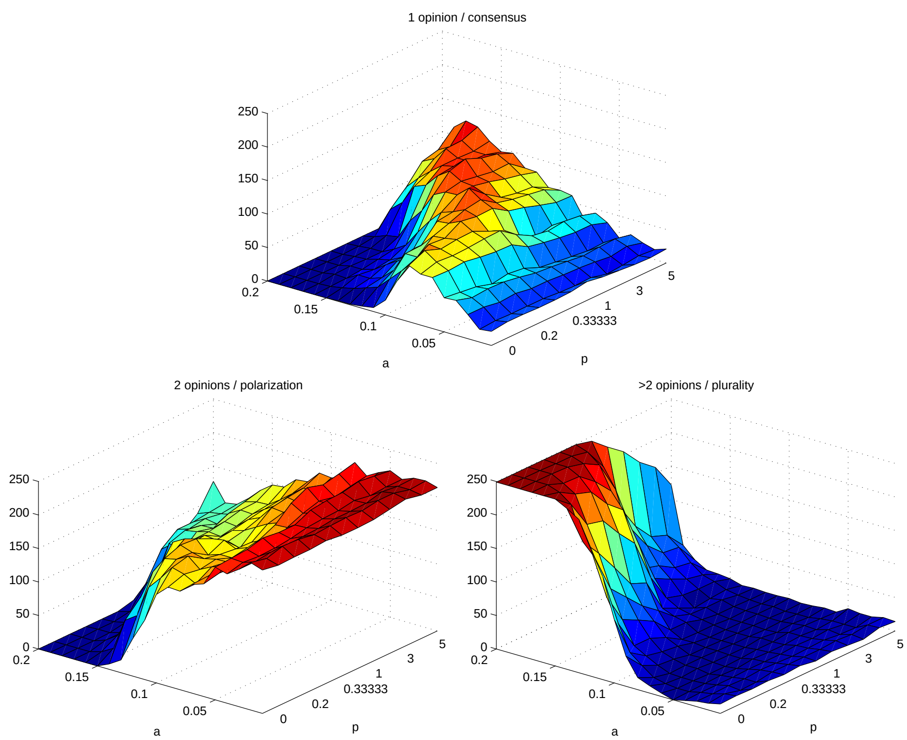

Figure 6: Distribution of different opinions in the convergence profile

Figure 6 shows the simulation results. We can use figure 5 as a legend for the graphics. The vertical axis shows, how many of the 250 starting profiles converge to consensus (top graphic), polarization (bottom left graphic) or plurality (bottom right graphic). Thus at every point (a, p) the values of all three graphics sum up to 250.

We see that the chance of reaching a consensus can be considerably raised. Even if we have no agents with medium bounds of confidence ($p = 0$) the chances to reach a consensus rises with rising range of heterogenity a until a value of about 0.08. For higher a the heterogenity gets too big and we get quickly into a situation were only plurality can be reached. If we have a greater width of the middle (a higher p) then higher chance for consensus can be reached. In our parameter field the best value ist $(a, p) = (0.18, 10)$. For this value the confidence vector is

$$\begin{bmatrix} 0.3800 \\ 0.3187 \\ 0.2768 \\ 0.2488 \\ 0.2303 \\ 0.2184 \end{bmatrix} \nearrow \begin{vmatrix} 0.2108 \\ 0.2062 \\ \vdots \\ 0.1938 \\ 0.1892 \end{vmatrix} \nearrow \begin{bmatrix} 0.1816 \\ 0.1697 \\ 0.1512 \\ 0.1232 \\ 0.0813 \\ 0.0200 \end{bmatrix}.$$

Thus, very few agents with high bounds of confidence together with very few agents with low bounds of confidence, raise the chance of reaching a consensus from 10 % up to 50 %.

But we have to notice that this way to choose confidence vectors is by far not the only possible way under the assumption of a constant confidence sum. There may be much more optimal kinds of heterogenity, that raise the chances for consensus even higher. Furthermore the optimal distribution of heterogenity may change if we change the sum of confidence.

So this paper should only be a first simulation and thought experiment for the way to explore opinion dynamics of agents with heterogeneous bounds of confidence.

4 Mathematical Results

The first aim of mahtematical analysis should be to prove the convergence of any process of opinion formation to a convergence profile. This work has not been completely done. Here, we will show a kind of provisional result, but concerning a more general case of opinion dynamics under changing confidence.

In our analysis we focus on the accumulation of confidence matrices.

For a process of opinion formation, it holds by iteration

$$x(t) = A(x(t-1))A(x(t-2)) \cdots A(x(1))A(x(0))x(0).$$

For abbreviaton we define for a series of matrices $(A(t))_{t \geq 0}$ the *accumulation* from time step t_0 to t_1 as

$$A(t_0, t_1) := A(t_1 - 1)A(t_1 - 2) \cdots A(t_0 + 1)A(t_0).$$

With definition $A(t) := A(x(t))$ we can write

$$x(t) = A(0, t)x(0).$$

Actually, we will not use all properties of confidence matrices, but only two properties, which hold for every confidence matrices $A(x)$ derived from an arbitrary opinion profile $x \in \mathbb{R}^m$:

1. $A(x)$ is row-stochastic. For every agent $i \in \underline{m}$ it holds $\sum_{j=1}^{m} a_{ij} = 1$ and $A(x)$ is non-negative.

2. The diagonal of $A(x)$ is positive. For every agent $i \in \underline{m}$ it holds $a_{ii} > 0$. (Every agent got a little bit of self-confidence.)

Property 2 leads us to this propositon: There exists a series of time steps $t_0 < t_1 < t_2 < \cdots$ such that $A(t_0, t_1), A(t_1, t_2), \ldots$ got the same zero-pattern.

A proof is shown in [7] (p. 4, Proposition 1, first part).

Now we define a confidence path. Let $A \in \mathbb{R}^{m \times m}$ be a non-negative matrix. There is a *confidence path* from $i \in \underline{m}$ to $j \in \underline{m}$, if there exist $i_1, \ldots, i_n \in \underline{m}$ such that $a_{ii_1} a_{i_1 i_2} \ldots a_{i_n j} > 0$. We write $i \longrightarrow j$.

We say i and j *communicate*, if $i \longrightarrow j$ and $j \longrightarrow i$.

For every quadratic non-negative matrix with positiv diagonal, we can sort the indices in disjunct classes of communicating indices. This means, all indices inside the class communicate, but no index is communicating with an index outside. Then we can differ between *essential* and *inessential* classes. From indices in essential classes there is no confidence path to any index outside of the class. From indices of inessential classes there is a confidence path to an index outside of the class. It is easy to see, that there must be at least one essential class. (For details see [8] pp. 9-14.)

The dividing in this classes depends only on the zero-pattern of the matrix, thus we get for every accumulation $A(t_k, t_{k+1})$ the same classes. Let $\mathcal{I}_1, \ldots, \mathcal{I}_p$ be the essential classes. And let $\mathcal{J}$ be the union of all inessential classes of indices in $A(t_k, t_{k+1})$. Thus, it holds $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p \cup \mathcal{J} = \underline{m}$.

If we renumber the indices of every matrix in our sequence of confidence matrices by $\mathcal{I}_1, \ldots, \mathcal{I}_p, \mathcal{J}$, then the matrix $A(t_k, t_{k+1})$ got (for every $k \in \mathbb{N}$) the following kanonical structure

$$\left[\begin{array}{ccc|c} A_{[\mathcal{I}_1,\mathcal{I}_1]} & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & A_{[\mathcal{I}_p,\mathcal{I}_p]} & 0 \\ \hline & A_{[\mathcal{I}_1 \cdots \mathcal{I}_p, \mathcal{J}]} & & A_{[\mathcal{J},\mathcal{J}]} \end{array}\right]. \tag{2}$$

Further, it can be shown that $A_{[\mathcal{I},\mathcal{I}]}$ is strictly positive for every essential class $\mathcal{I}$. (For a proof see [6] pp. 17-19.)

Notice that for every $t \geq t_0$ the accumulation $A(t_0, t)$ can be written in the same form as (2), by the rules of matrix multiplication.

Under very weak assumptions it holds for every essential class $\mathcal{I}$, that

$$\lim_{t \to \infty} A(t_0, t)_{[\mathcal{I},\mathcal{I}]} = K,$$

where K is a consensus matrix. A consensus matrix is a matrix with equal rows. (For any vector x the product Kx is a consensus vector. For more details see [6] pp. 21-26)

Furthermore, for the union of all inessential classes $\mathcal{J}$ it holds that

$$\lim_{t \to \infty} A(t_0, t)_{[\mathcal{J},\mathcal{J}]} = 0.$$

(Proof in [6] pp. 26-29.) This shows us, that the convergence opinion of an agent of an inessential class is in the long run only dependent on the weigths he gave to other agents and not at all to his starting opinion.

The positivity of the diagonal in every confidence matrix leads every process of opinion formation to some fixed structure of communicating classes. Some of them are essential and may find an internal consensus, some classes might be inessential, and the opinions of this agents depend only on the weights they gave to the agents in essential classes.

Some further research must be done to see totally clear how processes of opion formation under heterogenous bounds of confidence converges to a fixed opinion profile (what they do in every simulation).

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