Multidimensional Opinion Dynamics when Confidence Changes

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Abstract

A group of m agents is to find a common agreement about a certain issue. Consider this issue to be an n-dimensional vector of real numbers, for example the allocation of a fixed sum of money to n projects. Each agent has an opinion about the allocation, which he may revise. We model the process of opinion formation as a time-discrete dynamical system, in which every agent averages all opinions which are closed to his own opinion. Thus confidence structures may change. Mathematical analysis shows that every starting opinion distribution converges to a stable distribution. The driving force of this convergence is self-confidence. Further, we present some simulation results concerning $m = 150$ and $n = 3$, which give an insight into the dynamics of multidimensional opinion dynamics under bounded confidence.

1 The Model

Consider a group of m agents, each having an opinion about a certain issue. Consider that each opinion is a n-dimensional vector of real numbers. The group of agents is now to find an agreement. We suppose that each agent is willing to revise his opinion by taking the opinions of other competent agents into consideration. A competent agent in the view of one agent should be an agent with an opinion which is in a measurable way close to his own opinion. Further we suppose that all agents revise their opinions at the same time. The repitition of this simultaneous revising is what we call a process of opinion formation under bounded confidence.

The bounded confidence model with 1-dimensional opinions goes back to Krause/Hegselmann [6] and Dittmer [3]. In this paper we focus on multidimensionality of opinions.

We take an allocation problem as example. Consider the m agents are to allocate a fixed sum s of money or other resource among n projects. So an opinion is a non-negative vector, whose components sum up to s.

Lehrer/Wagner [8] (p. 107-112) have proved that every amalgamation of different opinions about an allocation can be modelled as a weighted arithmetic mean, if we got at least three projects and if we claim two not too strong axioms. One axiom is called zero unanimity. It means that the amalgamated opinion for project i must be zero, if all opinions about project i are zero. The second axiom is called label neutrality. If we permute the values of the projects in every opinion in the same way and do the amalgamation, then the amalgamated values are the same, as if we amalgamate first and permute then.

In the case of an allocation problem "revising an opinion" hence should be building a weighted arithmetic mean of all the opinions.

In the second half of this section, we model this process of opinion formation mathematically as a discrete dynamical system. In section 2, we show that every process of opinion formation converges to a fragmented but stable distibution of opinions. In section 3, we present some simulation results for an allocation problem with three projects.

For $m \in \mathbb{N}$ we define $\underline{m} := \{1, \dots, m\}$.

We call $X(t) \in \mathbb{R}^{m \times n}$ an *opinion profile* of m agents with an n-dimensional opinion at time step $t \in \mathbb{N}$. So every row of $X(t)$ is the opinion of one agent. In every column are the values of all agents for one parameter of the opinion. Although one opinion is a row-vector, we should not hesitate to imagine it as a point in $\mathbb{R}^n$.

For all agents we assume the same *range of confidence* $\varepsilon > 0$. Then we define the *confidence set* of agent i in opinion profile $X \in \mathbb{R}^{m \times n}$ as

$$I(X, i) := \{j\underline{m} \,|\, \|X_{[i,:]} - X_{[j,:]}\| \leq \varepsilon\}$$

for the euclidean norm.

We call $\{X \in \mathbb{R}^{1 \times n} \,|\, \|X_{[i,:]} - X\| \leq \varepsilon\}$ the *area of confidence* of agent $i \in \underline{m}$.

We assume that all agents distribute equal confidence weights between all agents with opinions which lie in their area of confidence. Thus we define a *confidence matrix* $A(X) \in \mathbb{R}^{m \times m}$ for an opinion profile $X \in \mathbb{R}^{m \times n}$ by

$$A(X)_{[i,j]} := \begin{cases} \frac{1}{\#I(X,i)} & \text{if } j \in I(X, i) \\ 0 & \text{otherwise.} \end{cases}$$

Now we can set up the *process of opinion formation* as a time-discrete dynamical system. Let $X(0) \in \mathbb{R}^{m \times n}$ be a starting opinion profile. The process of opinion formation is a series of opinion profiles $(X(t))_{t \geq 0}$ recursively defined through

$$X(t + 1) = A(X(t))X(t).$$

An opinion profile, which remains the same after iteration, we call *stabilized profile*. A *consensus* will be reached, if all rows of an opinion profile are equal. It is clear that a consensus is a stabilized profile.

We say that two agents $i, j \in \underline{m}$ *communicate* ($i \longleftrightarrow j$), if there exist agents $i_1, \dots, i_k \in \underline{m}$ such that i and i_1 trust each other ($a_{ii_1} > 0$, $a_{i_1 i} > 0$), and for all $l = 1, \dots, k-1$ the agents i_{l-1} and i_l trust each other ($a_{i_{l_1} i_l} > 0$, $a_{i_l i_{l-1}} > 0$) and finally i_k and j trust each other ($a_{i_k j} > 0$, $a_{j i_k} > 0$). It is easy to see that "$\longleftrightarrow$" is an equivalence relation. Thus, the set of agents $\underline{m}$ divides for every opinion profile into self-communicating classes $\mathcal{I}_1, \dots, \mathcal{I}_p$. Each agent communicates with every other agent in his class, but with no agent outside.

Thus, for a confidence matrix the set of indices $\underline{m}$ divides into self-communicating classes. Notice that the structure of self-communicating classes of indices depends only on the zero-pattern of the matrix.

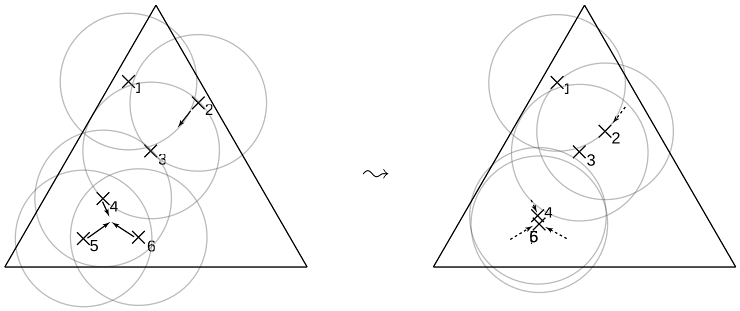

Figure 1: Example with 6 agents and 2-dimensional opinions

In figure 1, we show an example with an opinion profile of six agents with 3-dimensional opinions. Each opinion adds up to one. Thus every opinion lies in the unit simplex. We start with the cloud of the six opinions (the crosses) at the left side. The areas of confidence are circles around the opinions. In one time step every agent moves to the focal point of all opinions in its area of confidence. On the right side we see the cloud of opinions after one time step.

The confidence matrices for the two opinion profiles are

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

In the first cloud $\{1\}$ and $\{2, 3, 4, 5, 6\}$ are the self-communicating classes. After one time step the classes are $\{1, 2, 3\}$ and $\{4, 5, 6\}$. It happened a combination of splitting and joining of classes. This demonstrates the various possibilities of how confidence structures could change.

Nevertheless every process of opinion formation converges to a stabilized profile. This is what we will show in the next section.

2 Mathematical Analysis

In our analysis we focus on the accumulation of confidence matrices.

For a process of opinion formation, it holds by iteration

$$X(t) = A(X(t-1))A(X(t-2)) \cdots A(X(1))A(X(0))X(0).$$

For abbreviaton we define for a series of matrices $(A(t))_{t \geq 0}$ the *accumulation* from time step t_0 to t_1 as

$$A(t_0, t_1) := A(t_1 - 1)A(t_1 - 2) \cdots A(t_0 + 1)A(t_0).$$

With definition $A(t) := A(X(t))$ we can write

$$X(t) = A(0, t)X(0).$$

If we show that $\lim_{t\to\infty} A(0,t)$ converges to a constant matrix, then it is clear that $(X(t))_{t\geq 0}$ converges to a constant opinion profile.

Actually, we will not use all properties of confidence matrices, but only four properties, which hold for every confidence matrices $A(X)$ derived from an arbitrary opinion profile $X \in \mathbb{R}^{m\times n}$:

1. $A(X)$ is row-stochastic. For every agent $i \in \underline{m}$ it holds $\sum_{j=1}^{m} a_{ij} = 1$ and $A(X)$ is non-negative.
2. The diagonal of $A(X)$ is positive. For every agent $i \in \underline{m}$ it holds $a_{ii} > 0$. (Every agent got a little bit of self-confidence.)
3. Zero entries in $A(X)$ are symmetric and thus positivity of entries is also symmetric. For every two agents $i, j \in \underline{m}$ it holds $a_{ij} > 0 \Leftrightarrow a_{ji} > 0$. (Confidence is mutual.)
4. The lowest positive entry of $A(X)$ is greater than $1/m$.

At first, we will use property 2 for the following proposition.

Proposition 1 *Let $A(t) \in \mathbb{R}^{m\times m}$ be a series of matrices with positive diagonals for $t \in \mathbb{N}$, then there exists a series of time steps $t_0 < t_1 < t_2 < \cdots$ such that $A(t_0,t_1), A(t_1,t_2), \ldots$ got the same zero-pattern.*

Let $\mathcal{I}_1, \ldots, \mathcal{I}_p$ be the self-communicating classes of indices of the matrices $A(t_0,t_1), A(t_1,t_2), \ldots$. If we sort the indices of every matrix by simultaneous row and column permutations, then we got a block matrix with strictly positive blocks on the diagonal ($A(t_k,t_{k+1})_{[\mathcal{I}_i,\mathcal{I}_i]} > 0$ for all $k \in \mathbb{N}$,$i \in \underline{m}$) and zero-blocks at all other positions ($A(t_k,t_{k+1})_{[\mathcal{I}_i,\mathcal{I}_i]} > 0$ for all $k \in \mathbb{N}$ and $i, j \in \underline{m}$, $i \neq j$).

Proof. Notice that for any two non-negativ matrices with positive diagonals $A, B \in \mathbb{R}^{m\times m}$ it holds that every entry, which is positive in A or in B, is also positive in AB. Therefore, more and more positive entries in $A(0,t)$ appear monotonely increasing with rising t.

Thus, once there will be a time step t_0^* in which the maximum number of positive entries in $A(0,t)$ for all $t \in \mathbb{N}$ is reached. And it is clear that no matrix $A(t)$ with $t \geq t_0^*$ got a positive entry, where $A(0,t_0^*)$ has got a zero-entry.

If we now look at the series $(A(t))_{t\geq t_0^*}$, we find another time step t_1^*, such that $A(t_0^*,t_1^*)$ has reached again the maximum number of positive entries, but there are less or equal positive entries as in $A(0,t_0^*)$.

If we continue like this we got a series $A((t_i^*,t_{i+1}^*))_{i\geq 0}$ of accumulations in which positive entries vanish monotonely.

Thus, once there will be a time step $t_k^* =: t_0$ for which the minimum of positive entries is reached and so with $t_i := t_{i+k}^*$ we got the asserted series of time steps.

For the second half of the proposition, we first notice that it is clear that $A(t_k,t_{k+1})_{ij} = 0$ for all $i \in \mathcal{I}_l, j \in \mathcal{I}_q, k \in \mathbb{N}$ if $l \neq q$. The last thing to show is, that for every self-communicating class $\mathcal{I}_l$ it holds that $A(t_k,t_{k+1})_{[\mathcal{I}_l,\mathcal{I}_l]}$ is strictly positive.

For all $k \in \mathbb{N}$ the matrix $A(t_k,t_{k+1})_{[\mathcal{I}_l,\mathcal{I}_l]}$ is primitive (that means that one power is positive) because all indices are communicating and the diagonal is positive. Thus, there is one z such that

$$A(t_0,t_z)_{[\mathcal{I}_l,\mathcal{I}_l]} = A(t_{z-1},t_z)_{[\mathcal{I}_l,\mathcal{I}_l]} \cdots A(t_0,t_1)_{[\mathcal{I}_l,\mathcal{I}_l]}$$

is strictly positive, because the primitivity property depends only on the zero pattern of a matrix, which is equal in $A(t_k, t_{k+1})_{[\mathcal{I}_l,\mathcal{I}_l]}$ for every $k \in \mathbb{N}$.

Thus, $A(t_k, t_{k+1})_{[\mathcal{I}_l,\mathcal{I}_l]}$ must be strictly positive for all k because otherwise, there were less positive entries than in later accumulations, which is a contradiction to the minimality of positive entries. □

We observe that the agents tend to split into a kind of self-communitating classes, which is only forced by the existence of some self-confidence for every agent.

The next proposition is a result about the convergence of accumulations of row-stochastic matrices to a consensus matrix. A consensus matrix is a row-stochastic matrix with equal rows.

Proposition 2 *Let $A(t) \in \mathbb{R}^{m\times m}$ be a series of row-stochastic matrices for $t \geq 0$ and let $\delta_t > 0$ be a series with $\sum_{t=0}^{\infty} \delta_t = \infty$. If it holds for all t that*

$$\min_{i,j} \sum_{k=1}^{m} \min\{a(t)_{ik}, a(t)_{jk}\} \geq \delta_t$$

then there exists a consensus matrix K such that

$$\lim_{t\to\infty} A(0,t) = K.$$

Proof. (Only sketched) If we interprete the rows of a matrix $A(t) \in \mathbb{R}^{m\times m}$ as vectors in $\mathbb{R}^m$ then we can define the row-diameter $\mathrm{d}(A(t))$ as the maximum distance of two rows.

Multiplication from the left with a row-stochastic matrix ($A(t+1)$ in our setting) shrinks the row-diameter in this way

$$\mathrm{d}(A(t+1)A(t)) \leq \left(1 - \min_{i,j} \sum_{k=1}^{m} \min\{a_{ik}(t+1), a_{jk}(t+1)\}\right) \mathrm{d}(A(t)).$$

For a proof see [9] (p. 22-23, Satz 2.4.7). This is a more dimensional version of the so-called shrinking lemma, seen for example in [7].

Thus the row-diameter shrinks to zero by iteration, because of the conditions about δ_t. And this forces all rows to converge to the same fixed values.

For details see [9] (p. 24-25, Satz 2.4.11) or arguments in [7] (p. 229-230, Theorem 1) which can be easily modified for the more-dimensional case. □

If we look at the series of accumulations $(A(t_i, t_{i+1}))_{i\geq 0}$ in proposition 1, sorted by their self-communicating classes, then we can hope for a convergence of every block on the diagonal, which is strictly positive by proposition 2, to a consensus matrix. We only have to find a series δ_i, with $\sum_{i=0}^{\infty} \delta_i = \infty$, such that the lowest positive entry of $A(t_i, t_{i+1})$ is greater than δ_i.

Proposition 3 *Let $A(t) \in \mathbb{R}^{m\times m}$ be a series of confidence matrices for $t \geq 0$. Then it holds for every two time steps $t_0 < t_1$ that the lowest positive entry of $A(t_0, t_1)$ is greater than $(1/m)^{m^2-m+2}$.*

Proof. (Only sketched) Let $t_0 < t_1$ and $N := t_1 - t_0$. Let $\mu(A)$ be the lowest positive entry of the non-negative matrix A. It is easy to see that

$$\mu(A(t_0, t_1)) = \mu(A(t_1 - 1) \cdots A(t_0)) \geq \mu(A(t_1 - 1)) \cdots \mu(A(t_0)).$$

With property 4 it holds $\mu(A(t_0, t_1)) \geq (1/m)^N$.

It holds that the lowest positive entry of a product $A(t+1)A(t)$ can only be lower than the lowest positive entry of $A(t)$, if $A(t+1)A(t)$ got at least one zero entry less. For a proof see [9] (p. 28-29, Lemma 2.5.7). The proof uses property 3, the symmetry of the zero entries in every matrix.

If we want the lowest positive entry of $A(t_0, t_1)$ to sink below $(1/m)^{m^2-m+2}$, then $m^2 - m + 1$ zeros have to disappear while we are accumulating, but then $A(t_0)$ must have at least $m^2 - m + 1$ zeros and thus can not have a positive diagonal. □

Now we can put the three propositions togehter.

Theorem 1 *Let $A(t) \in \mathbb{R}^{m\times m}$ be a series of confidence matrices for $t \geq 0$. Then it exists a time step t_0 and pairwise disjunct classes $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p = \underline{m}$ such that*

$$\lim_{t\to\infty} A(0,t) = \begin{bmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_p \end{bmatrix} A(0, t_0),$$

and $K_1, \dots, K_p$ are quadratic consensus matrices in the sizes of $\mathcal{I}_1, \dots, \mathcal{I}_p$. (For this we must think that every matrix is sorted by $\mathcal{I}_1, \dots, \mathcal{I}_p$. This is possible without loss of generality.)

Proof. From proposition 1, we get the time step t_0 and further time steps $t_1, t_2, \dots$ so that we get a constant distribution into communicating classes of indices for the accumulations $A(t_i, t_{i+1})$ with $i \geq 0$.

Because of this we can compute for every $t \geq t_0$ and $q \in \underline{p}$ the accumulation $A(t_0, t)_{[\mathcal{I}_q, \mathcal{I}_q]}$ only form the matrices $A(t)_{[\mathcal{I}_q, \mathcal{I}_q]}$.

Proposition 1 says also that the matrices $A(t_i, t_{i+1})_{[\mathcal{I}_q, \mathcal{I}_q]}$ are strictly positive for all $i \in \underline{m}$ and $q \in \underline{p}$. And from proposition 3, we get that their lowest entry is greater than $(1/m)^{m^2-m+2}$.

Thus for every $q \in \underline{p}$ holds that $A(t_0, t)_{[\mathcal{I}_q, \mathcal{I}_q]}$ converges for $t \to \infty$ to a consensus matrix by proposition 2. □

Thus, it holds that every process of opinion formation converges in the way that each of the self-communicating classes finds a consensus. Thus, a consensus among all agents can be reached, if and only if, every self-communicating class reaches the same consensual values. This is especially possible, if the agents remain in only one self-communicating class.

In the case of bounded confidence, the stabilized profile will be reached in finite time, because once there will be a time step, in which the distance of all opinions, which converge to the same value, is below ε.

3 Simulation results

We simulate 150 agents with an allocation problem about three projects. Thus an opinion should consist of three positive real numbers adding up to one. Thus every opinion is in the 3-dimensional unit simplex, which is the convex hull of $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subset \mathbb{R}^3$.

Figure 2 shows an example with a randomly choosen starting opinion profile, where opinions are distributed evenly in the simplex. The range of confidence is $\varepsilon = 0.25$. On the left side we see the 150 opinions as black points. In the middle we see the opinion cloud at time step $t = 3$, and at the right side we see the opinion profile at $t = 6$, which is already the stabilized profile, which must be reached by the theorem of section 2. The crosses stand for the starting distribution and the gray lines stand for the orbits the opinions walk.

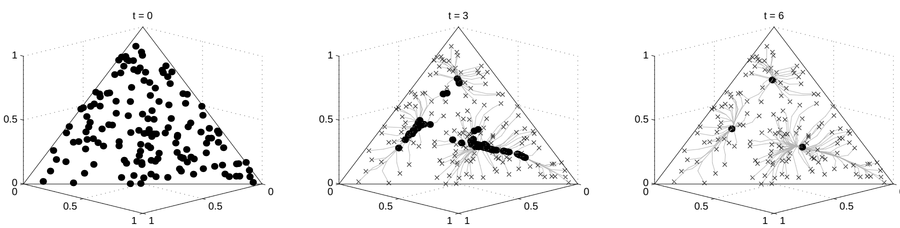

Figure 2: 150 agents, $\varepsilon = 0.25$, opinion cloud at time steps $t = 0, 3, 6$

Here is a short description of the driving forces of the opinion dynamic: At first all the opinions being close to the border of the simplex will move a little bit in the direction of the middle because outside of the simplex are no opinions. Opinions in the middle of the simplex would not move in a particular way. Thus, at the first time steps the cloud of opinions shrinks, but also there will be a an aggregation of opinions at the border of the cloud. At the three edges of the shrinked cloud the concentration will be most dense. An aggregation of opinions attracts other opinions in its range. Thus, there is a quick attractions at three points each corresponding to one edge of the simplex.

We are now interested to see, how the number of different opinions after stabilization and their position depends on the range of confidence ε.

The complexity of the behavior can be demonstrated by figure 3. For this figure we always use the same start profile as in figure 2 but with different values for ε form 0.15 to 0.3. On the vertical axis we assign the number of different opinions after stabilization. As we aspect this number decreases as confidence rises, but this is not monotone.

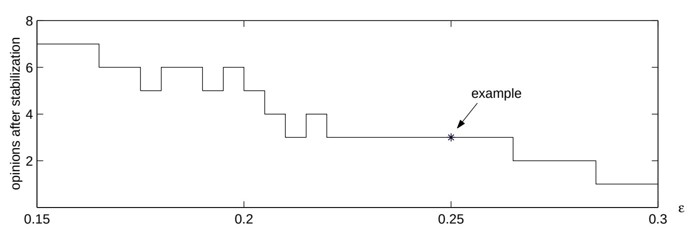

Figure 3: Number of opinions after stabilization for rising ε

Thus, we switch to a statistical analysis. We take 1000 random start profiles, each with 150 opinions which are evenly distributed. We choose a range of confidence ε and compute all 1000 stabilized profiles. We repeat this for ε-values from 0.15 to 0.3.

In figure 4, we show visualizations of the computed data for $\varepsilon = 0.17, 0.23, 0.27$. On the left side we see the simplex divided into 210 small simplices. We count for every small simplex how many of the 150.000 stabilized opinions lie in it.

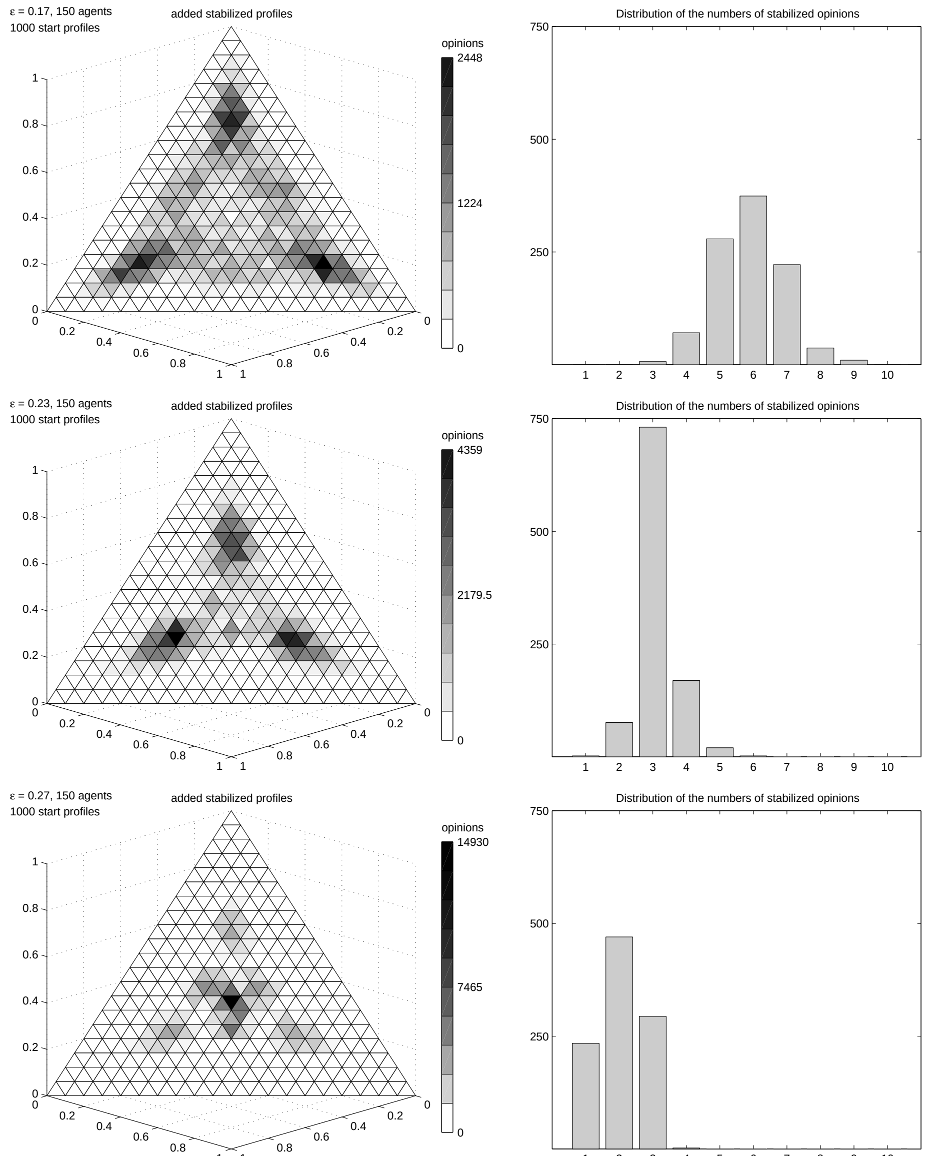

Figure 4: Added stabilized profiles for $\varepsilon = 0.17, 0.23, 0.27$

The intensity of a small simplex depends on the number of opinions it contains. A gray-scale axis with explicit numbers is at the right side of the graphic. On the left side we can see areas were the stabilized opinions aggregate, this can be interpreted as a kind of probability. But we can not see explicit how many different stabilized opinions remain. For this information, we got the graphics on the right side. It shows the distribution of the number of different stabilized opinions for the 1000 start profiles.

In figure 5, we finally show this distribution of the number of different stabilized opinions for ε from 0.15 to 0.3 in 0.01-steps. The dark rows are the three distibutions of figure 4. Along the gray rows the heights of all bars sum up to 1000.

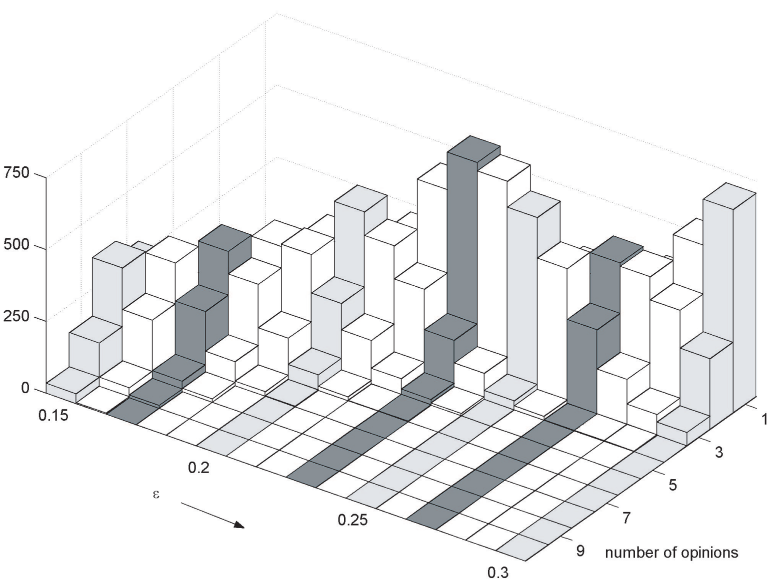

Figure 5: Distribution of the number of opinions after stabilization

A short description of the results of the statistical analysis: We see that the number of different stabilized opinions sinks with rising ε, as expected. For $\varepsilon = 0.17$ the number varies around six. We could call this *plurality*. For $\varepsilon = 0.23$ the number varies around three but variation is very much lower than before, nearly 75% is three. By the graphic of added stabilized profiles we see that the positions of the three opinions are quite predictable, because we see three dark areas. We could call this *predictable polarization* into three opinions. For $\varepsilon = 0.27$ the number varies around two and variation is higher than before. By the graphic of added stabilized profiles we can not clearly detect the position of the two opinions. We could call this *unpredictable polarization* into two opinions. We can explain this, if we think of the three characteristic positions were the opinions aggregate after the first time steps of a process. Then two of this aggregations unite and one stands alone, so we got two different stabilized opinions, but we cannot predict which aggragation remains alone. If we raise ε up to 0.29 then the *consensus* gets the dominant stabilized profile, as we can see in figure 5. Figure 5 shows also explicit, that polarization on three opinions is a dominant behavior for a much greater range of ε, than polarization into other numbers of opinions. Only the consensus will clearly reach even higher domination, if ε is raised further.

For information about the programming see [9] or contact the author.

4 Summary

From the mathematical analysis we can derive the strong influence that self-confidence result into the stabilization of processes of opinion formation. The tendency to stabilize is not depended on the special properties of opinion dynamics under bounded confidence, but can be proved for every process of opinion formation, where agents do a kind of averaging of some other opinions (property 1), where every agent got a little bit of self-confidence (property 2) and where confidence is mutual (property 3). We also need property 4, but this should mainly prevent the convergence of the series of confidence matrices, to matrices which do not fulfill properties 1-3.

With simulations we examined the dynamic behavior of a 3-dimensional process of opinion formation under bounded confidence. The possible opinions were allocations of the sum of one to three projects, and the starting opinions were distributed evenly in the opinion space. We observed for many values of the range of confidence a great tendency of polarization into three opinions, each of this three opinions gives a great part to one project and a small parts to the other two agents (approximately [.6 .2 .2]). More colloquial, the agents divide into three groups each group favors one project. For higher ranges of confidence the tendency goes to polarization into two opinions. One of them favors one project, and the other gives a very small part to one project and divides the big rest evenly between the two other projects (approximately [.4 .4 .2]). But we can not predict, which two of the three groups unite. For even higher ranges of confidence the tendency goes to consensus, which is always an opinion were every project gets nearly a third.

References

[1] Samprit Chatterjee and Eugene Seneta. Towards consensus: Some convergence theorems on repeated averaging. *J. Appl. Prob.*, 14:159–164, 1977.

[2] M. H. DeGroot. Reaching a consensus. *Journal of American Statistical Association,*, 69(345):118–121, 1974.

[3] Jan Christian Dittmer. *Diskrete nichtlineare Modelle der Konsensbildung.* Diplomarbeit, Universität Bremen, April 2000.

[4] F. Amblard G. Deffuant, D. Neau and G. Weisbuch. How can extremism prevail? a study based on the relative agreement interaction model. *Journal of Artificial Societies and Social Simulation*, 5(4), 2002. `http://www.soc.surrey.ac.uk/JASSS/5/4/1.html`.

[5] F. Amblard G. Deffuant, J. P. Nadal and G. Weisbuch. Mixing beliefs among interacting agents. *Advances in Complex Systems*, 3:87–98, 2000.

[6] Rainer Hegselmann and Ullrich Krause. Opinion dynamics and bounded confidence, models, analysis and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3), 2002. `http://www.soc.surrey.ac.uk/JASSS/5/3/2.html`.

[7] Ullrich Krause. A discrete nonlinear and non-autonomous model of consensus formation. In S. Elyadi, G. Ladas, J. Popenda, and J. Rakowski, editors, *Communications in Difference Equations*, pages 227–236. Gordon and Breach Pub., Amsterdam, 2000.

[8] Keith Lehrer and Carl Wagner. *Rational Consensus in Science and Society.* D. Reidel Publishing Company, Dordrecht, Holland, 1981.

[9] Jan Lorenz. *Mehrdimensionale Meinungsdynamik bei wechselndem Vertrauen.* Diplomarbeit, Universität Bremen, February 2003.

[10] Eugene Seneta. *Non-negative Matrices.* George Allen & Unwin Ltd., London, 1973.